15.2 Double Integrals over General Region

Last time:

For the rectangular region, *R*:

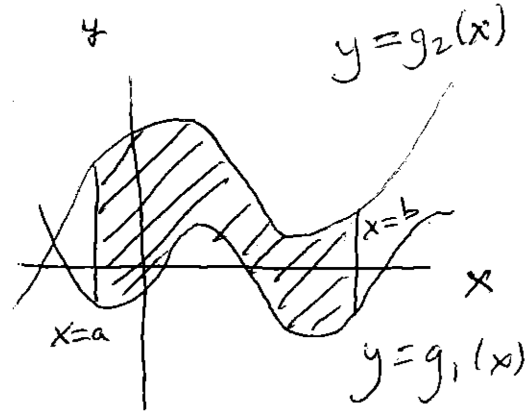
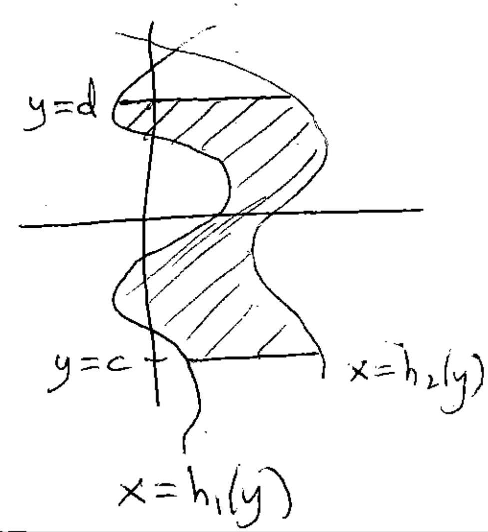
$$a \leq x \leq b \ , \qquad c \leq y \leq d$$

we learned

$$\iint_R f(x,y)\,dA$$

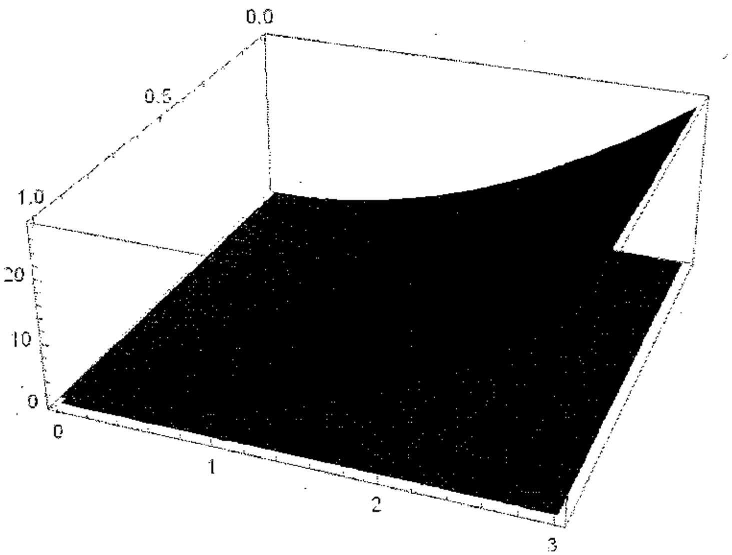
$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

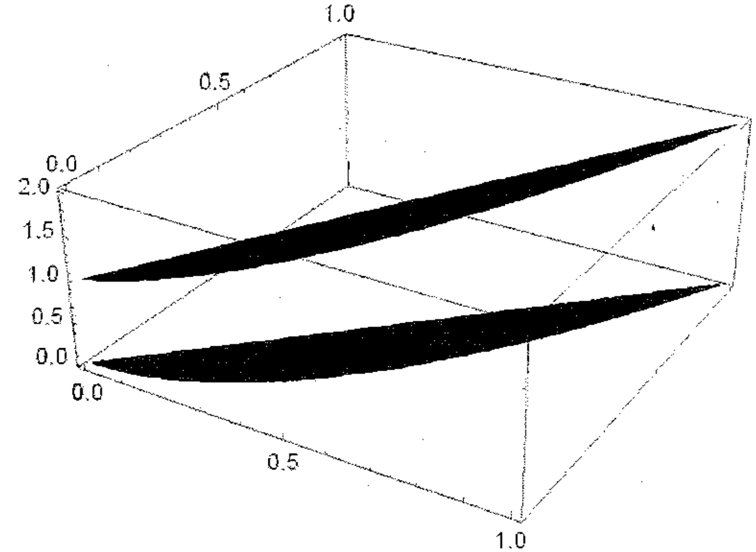
Type 1 (Top/Bot)	Type 2 (Left/Right)
	
For all *x* in the range, $a \leq x \leq b$, we have $g_1(x) \leq y \leq g_2(x)$	For all *y* in the range, $c \leq y \leq d$, we have $h_1(y) \leq x \leq h_2(y)$
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$

In 15.2, we discuss regions other than rectangles.

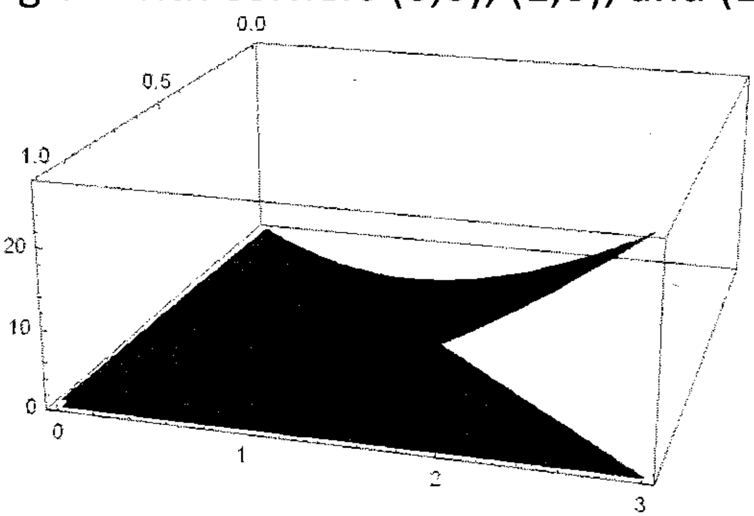
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



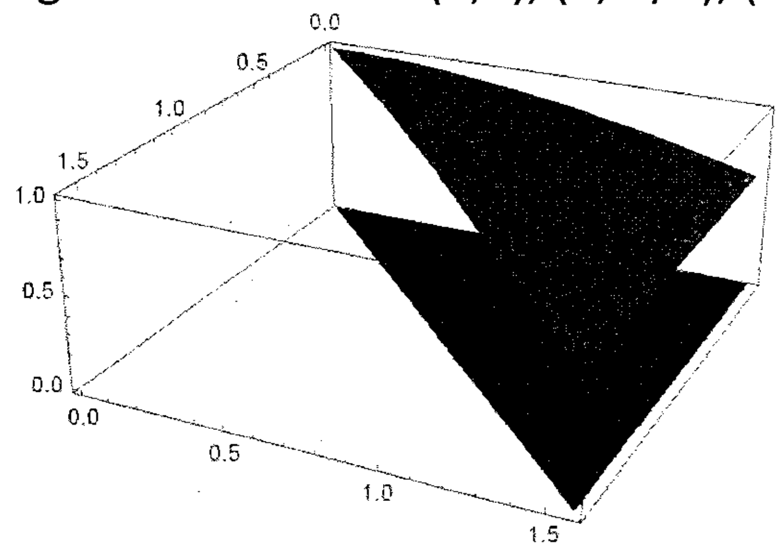
The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = x + 3y^2$ over the triangular region with corners (0,0), (1,0), and (1,3).



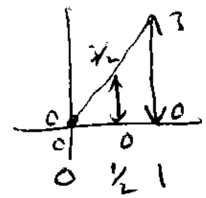
The surface $z = \sin(y)/y$ over the triangular region with corners (0,0), $(0, \pi/2)$, $(\pi/2, \pi/2)$.

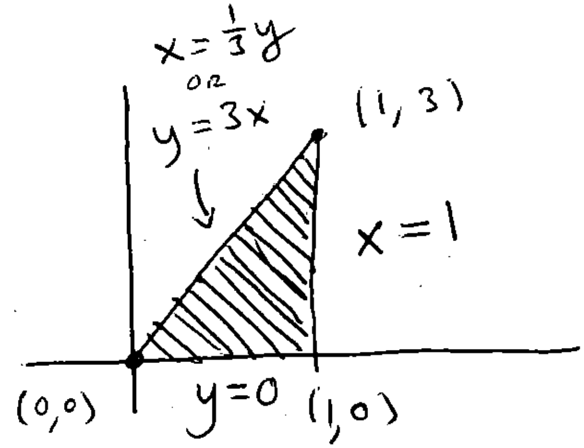


Examples:

1. Let D be the triangular region in the xy-plane with corners (0,0), (1,0), (1,3).

Evaluate $\iint_D x + 3y^2 dA$





OPTION 1: FIX x FIRST

$0 \le x \le 1$

$\Rightarrow 0 \le y \le 3x$

Ex)

x	
0	$0 \le y \le 0$
1	$0 \le y \le 3$
$\frac{1}{2}$	$0 \le y \le \frac{3}{2}$

$$\int_0^1 \left(\int_0^{3x} x + 3y^2 \, dy \right) dx$$

$$\int_0^1 xy + y^3 \Big|_0^{3x} dx$$

$$\int_0^1 3x^2 + 27x^3 \, dx$$

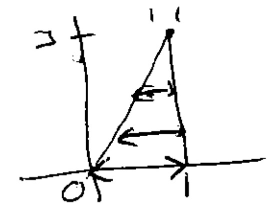
$$= x^3 + \frac{27}{4} x^4 \Big|_0^1$$

$$= 1 + \frac{27}{4} = \boxed{\frac{31}{4}} = 7.75$$

OPTION 2: FIX y FIRST

$0 \le y \le 3$

$\Rightarrow \frac{1}{3} y \le x \le 1$



Ex)

y	
0	$0 \le x \le 1$
3	$1 \le x \le 1$
1	$\frac{1}{3} \le x \le 1$
2	$\frac{2}{3} \le x \le 1$

$$\int_0^3 \left(\int_{\frac{1}{3}y}^1 x + 3y^2 \, dx \right) dy$$

$$= \int_0^3 \frac{1}{2} x^2 + 3y^2 x \Big|_{\frac{1}{3}y}^1 dy$$

$$= \int_0^3 \frac{1}{2} + 3y^2 - \frac{1}{18} y^2 - y^3 \, dy$$

$$= \frac{1}{2} y + y^3 - \frac{1}{54} y^3 - \frac{1}{4} y^4 \Big|_0^3$$

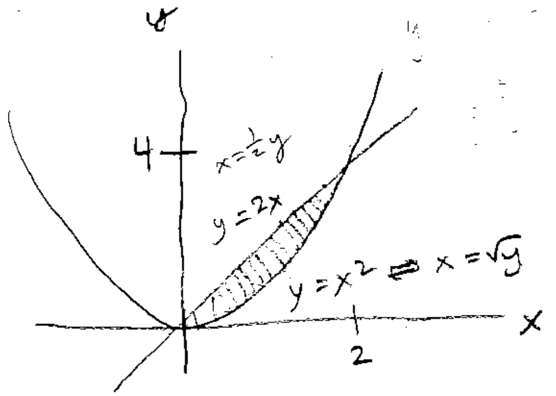
$$= \frac{3}{2} + 27 - \frac{1}{2} - \frac{81}{4} = 1 + \frac{108}{4} - \frac{81}{4}$$

$$= 1 + \frac{27}{4} = \boxed{\frac{31}{4}} = 7.75$$

2. Find the volume of the solid bounded by the surfaces $z = x + 1, y = x^2,$ $y = 2x, z = 0.$

Step 1 $\iint_D x+1 \, dA$

STEP 2



$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$x = 0$ or $x = 2$

OPTION 1: $0 \le x \le 2$

$\Rightarrow x^2 \le y \le 2x$

$$\int_0^2 \left(\int_{x^2}^{2x} x+1 \, dy \right) dx$$

$$= \int_0^2 xy + y \Big|_{x^2}^{2x} dx$$

$$= \int_0^2 2x^2 + 2x - x^3 - x^2 \, dx$$

$$= \frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4 \Big|_0^2$$

$$= \frac{8}{3} + 4 - 4 = \boxed{8/3}$$

OPTION 2: $0 \le y \le 4$

$\frac{1}{2}y \le x \le \sqrt{y}$

$$= \int_0^4 \left(\int_{\frac{1}{2}y}^{\sqrt{y}} x+1 \, dx \right) dy$$

$$\int_0^4 \left(\frac{1}{2}x^2 + x \Big|_{\frac{1}{2}y}^{\sqrt{y}} \right) dy$$

$$\int_0^4 \cancel{\frac{1}{2}y} + \sqrt{y} - \frac{1}{8}y^2 - \cancel{\frac{1}{2}y} \, dy$$

$$= \frac{2}{3}y^{3/2} - \frac{1}{24}y^3 \Big|_0^4$$

$$= \frac{16}{3} - \frac{1}{6}16$$

$$= \frac{16}{3} - \frac{8}{3} = \boxed{8/3} \approx 2.\overline{6}$$

3. Draw the region of integration for

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{y}\, dy\, dx$$

then switch the order of integration.

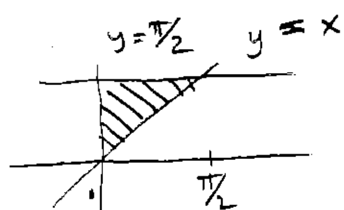
GIVEN $\quad 0 \le x \le \pi/2$

$\Rightarrow x \le y \le \pi/2$

STEP 1: TOP/BOT or LEFT/RIGHT ?

BOT $\to y = x$

TOP $\to y = \pi/2$



STEP 2: TICKMARKS $\quad x = 0$ to $x = \pi/2$

YOU MUST DRAW THE PICTURE!!!

THAT IS THE BEST WAY TO DO IT!

REVERSE ORDER

$0 \le y \le \pi/2$

$\Rightarrow 0 \le x \le y$

$$\int_0^{\pi/2}\left(\int_0^{y} \frac{\sin(y)}{y}\, dx\right) dy$$

$$\int_0^{\pi/2} \frac{\sin(y)}{y}\, x \Big|_0^y \, dy$$

$$\int_0^{\pi/2} \frac{\sin(y)}{y} \cdot y - 0 \, dy$$

$$\int_0^{\pi/2} \sin(y)\, dy$$

$$-\cos(y)\Big|_0^{\pi/2}$$

$$-\underbrace{\cos(\pi/2)}_{0} - -\underbrace{\cos(0)}_{1}$$

$$= \boxed{1}$$

4. Switch the order of integration for

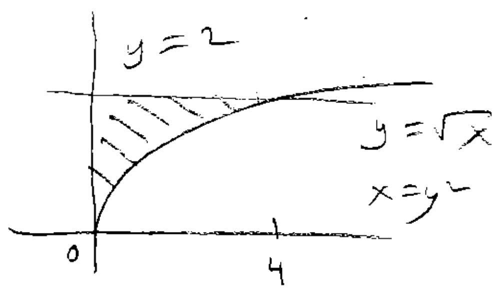
$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy dx$$

GIVEN $0 \le x \le 4$

$\Rightarrow \sqrt{x} \le y \le 2$

BOT $\rightarrow$ $y = \sqrt{x}$

TOP $\rightarrow$ $y = 2$



$0 \le y \le 2$

$0 \le x \le y^2$

$$\int_0^2 \int_0^{y^2} \sin(y^3)\, dx\, dy$$

$$\int_0^2 \sin(y^3)\, x \Big|_0^{y^2} dy$$

$$\int_0^2 y^2 \sin(y^3) - 0\, dy$$

$$\int_0^8 y^2 \sin(u) \frac{1}{3y^2}\, du$$

$u = y^3$

$du = 3y^2 dy$

$\frac{1}{3y^2} du = dy$

$$\frac{1}{3}\int_0^8 \sin(u)\, du$$

$$-\frac{1}{3}\cos(u)\Big|_0^8$$

$$-\frac{1}{3}\cos(8) - -\frac{1}{3}\cos(0)$$

$$\boxed{-\frac{1}{3}\cos(8) + \frac{1}{3}} \approx 0.381833$$

Setting up a problem given in "words":

1. Find integrand

Solve for "z" anywhere you see it.

If there are two z's, then set up two double integrals (subtract at end).

2. Region?

Graph the region in the *xy*-plane.

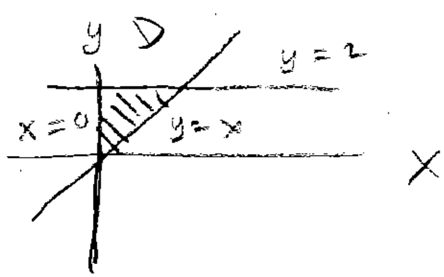
a) Graph all given *x* and *y* constraints.

b) And find the xy-curves where the surfaces (the z's) intersect.

Examples (directly from HW):

HW 15.2: Find the volume enclosed by $z = 4x^2 + 4y^2$ and the planes $x = 0$, $y = 2$, $y = x$, and $z = 0$.

$$\iint_D 4x^2 + 4y^2 \, dA$$



HW 15.3:

Find the volume below $z = 18 - 2x^2 - 2y^2$ and above the xy-plane. ← $z = 0$

$$\iint_D 18 - 2x^2 - 2y^2 \, dy$$

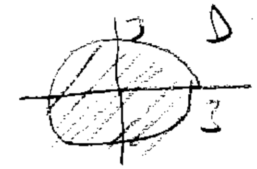
NO xy-CONSTRAINT GIVEN???

INTERSECT!

$$18 - 2x^2 - 2y^2 \overset{?}{=} 0$$

$$\Rightarrow \quad 18 = 2x^2 + 2y^2$$

$$\Rightarrow \quad 9 = x^2 + y^2$$



HW 15.3:

$z = \pm\sqrt{22 + x^2 + y^2}$

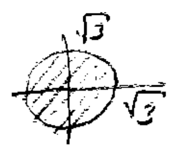
Find the volume enclosed by
$-x^2 - y^2 + z^2 = 22$ and $z = 5$.

$$\iint_D 5 \, dA \quad \text{AND} \quad \iint_D \sqrt{22 + x^2 + y^2} \, dA$$

INTERSECT

$(5)^2$

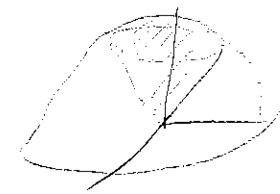
$$-x^2 - y^2 + 25 = 22$$
$$3 = x^2 + y^2$$



HW 15.3:

Find the volume above the upper cone
$z = \sqrt{x^2 + y^2}$ and
below $x^2 + y^2 + z^2 = 81$

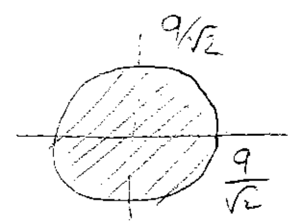
$z = \pm\sqrt{81 - x^2 - y^2}$



$$\iint_D \sqrt{81 - x^2 - y^2} \, dA \quad \text{AND} \quad \iint_D \sqrt{x^2 + y^2} \, dA$$

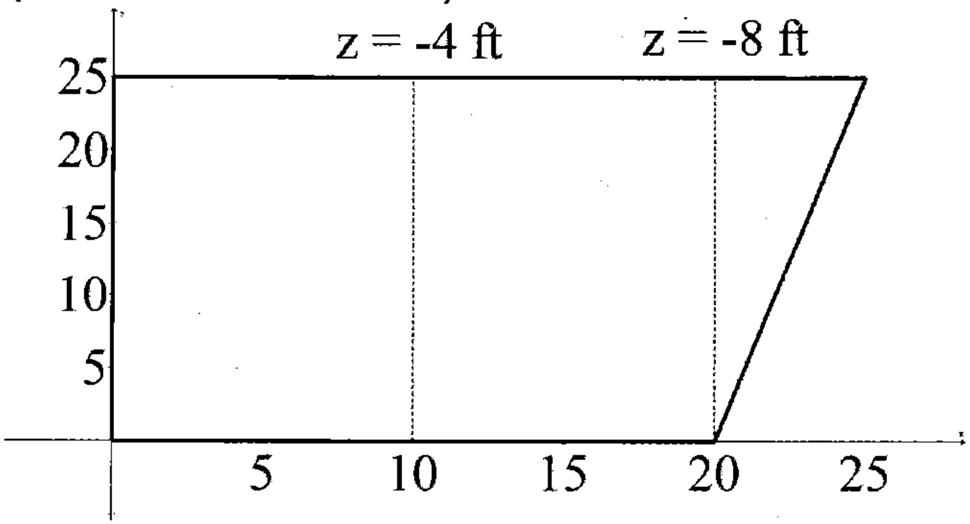
INTERSECT

$$x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 81$$
$$\Rightarrow x^2 + y^2 + x^2 + y^2 = 81$$
$$\Rightarrow 2x^2 + 2y^2 = 81$$
$$\Rightarrow x^2 + y^2 = \frac{81}{2}$$



An applied problem:

Your swimming pool has the following shape (viewed from above)



The bottom of the pool is a plane with depths as indicated (the pool gets deeper in a linear way from left-to-right)

Solution:

1. Surface?

Slope in y-direction = 0

Slope in x-direction = -4/10 = -0.4

Also the plane goes through (0, 0, 0)

Thus, the plane that describes the bottom of the pool is: **z = -0.4x + 0y**

2. Region?

The line on the right goes through (20,0) and (25,25), so it has slope = 5 and it is given by the equation

y = 5(x-20) = 5x – 100

or **x = (y+100)/5 = 1/5 y + 20**

The best way to describe this region is by thinking of it as a left-right region.

On the left, we have x = 0

On the right, we have x = 1/5 y + 20

Therefore, we have

$$\int_0^{25} \left(\int_0^{\frac{1}{5}y+20} -0.4\, x\, dx \right) dy = -741.\bar{6} \ \ \text{ft}^3$$